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MODEL OF BENDING OF A HYDROSTATICALLY COMPRESSED SHELL NEAR ITS STABILITY THRESHOLD

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A simplified model of bending dynamics of a hydrostatically compressed thin shell near the threshold of stability of its form is constructed within the framework of the nonlinear theory of elasticity. Conditions of existence and explicit expressions for spatially localized perturbations and patterns composed of dents on the shell surface, which are "precursors" of subsequent changes in the shell form, are found.

Key words: *shell, nonlinear elasticity, stability, soliton.*

A circular cylindrical shell compressed from the outer side of the surface by a high-pressure fluid loses stability of its form at a certain value of the external pressure ("collapse" or nonlinear buckling occurs). As a result, bumps and dents extended along the shell generatrix appear on the shell surface; in the shell cross section, these structures alternate [1]. Deformations of the thin shell remain elastic and, as a whole, are consistent with its geometric bending [2]. Therefore, the initial stage of changes in the shell form can be described within the framework of the nonlinear theory of elasticity [3–5]. Equations of the nonlinear theory of elasticity take into account not only the geometric nonlinearity of the problem caused by nonlinearity of the strain tensor, but also the physical nonlinearity characterizing the material properties and described by the highest invariants of the strain tensor in the expansion of the expression for nonlinearly elastic energy of the medium. Correct allowance of the highest invariants is of principal importance, because the emergence of nonlinearity effects leads to localization of shell bendings. In the final analysis, formation of spatially localized patterns from dents on the shell surface at the initial stage of changes in its form is the result of interaction of nonlinearity and dispersion effects.

One important feature of the problem is the absence of dispersion terms in the original equations of the nonlinear theory of elasticity. In simplified models of shells, dispersion terms appear owing to elimination of the "fast" variable characterizing the nonuniformity of strains along the normal to the shell surface.

There are some novel methods [6] that allow obtaining simplified models, based on proven equations of the nonlinear theory of elasticity without using *a priori* hypotheses and with controlled accuracy in terms of small parameters characterizing the shell size, magnitude of external stress, space and time scales of deformations, and geometric and physical nonlinearity of the problem. Such methods reveal latent dynamic symmetry of the problem; therefore, the simplified equations are universal and close to integrated models, which allows their solutions to be analyzed in detail by methods of the advanced theory of solitons. Nevertheless, the nonlinearly elastic dynamics of shells near their stability thresholds and the possibilities of its approximation by integrated models have not been studied yet.

A variant of the reductive perturbation theory suitable for solving nonlinear boundary-value problems, where the final surface of the deformed shell is not known in advance and is found in the course of solving the problem, is proposed in the present paper. The expression for the initial nonlinearly elastic energy of the material is presented in the form of an expansion over all strain-tensor invariants admitted by medium symmetry. The method proposed allows automatic selection of invariants from equations of the nonlinear theory of elasticity and contributions of

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these invariants necessary for constructing a simplified model. It turned out that different invariants are needed for solving different problems; as a result, different models are constructed.

1. Constitutive Equations. Let $\boldsymbol{i}_k$ $(k = 1, 2, 3)$ be the basis unit vectors of the Cartesian coordinate system (the vector $\boldsymbol{i}_1$ is directed along the mid-surface of the circular cylindrical shell). The position of the material particle of the nondeformed shell is described by the radius vector $\boldsymbol{r} = x^s \boldsymbol{i}_s$. Summation it performed over repeated indices. The indices denoted by Latin letters take the values 1, 2, and 3.

For the further analysis, it is convenient to pass from the Cartesian coordinate system to a curvilinear coordinate system, which is close to a cylindrical coordinate system and, hence, takes better account of the problem symmetry. The position of the material particle of the nondeformed shell is characterized by the coordinates

$$y^1 = x^1, \qquad y^2 = \frac{R}{2i} \ln \frac{x^2 + ix^3}{x^2 - ix^3}, \qquad y^3 = \sqrt{(x^2)^2 + (x^3)^2}$$

(R is the radius of the mid-surface of the nondeformed shell and i is the imaginary unit). The vectors of the local reference point and the components of the metric tensor have the form

$$\boldsymbol{e}_k = \frac{\partial x^s}{\partial y^k}\, \boldsymbol{i}_s, \qquad g_{ik} = \frac{\partial x^s}{\partial y^i}\, \frac{\partial x^s}{\partial y^k} = \operatorname{diag}\Big(1, \Big(\frac{y^3}{R}\Big)^2, 1\Big).$$

Such a choice of the local reference point yields standard equations of the shell theory.

During shell deformation, the material particle with the radius vector $\boldsymbol{r}$ acquires a displacement $\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{v}$, where $\boldsymbol{v} = v^s(\boldsymbol{y}, t)\boldsymbol{e}_s$. The Lagrangian strain tensor is determined by the relation

$$E_{km} = \frac{1}{2}\Big(\nabla_k v_m + \nabla_m v_k + \nabla_k v^s \nabla_m v_s\Big), \qquad \nabla_k v^s = \frac{\partial}{\partial y^k}\, v^s + \Gamma^s_{kp} v^p. \tag{1}$$

Here $\nabla_k v^s$ is the absolute (covariant) derivative of the displacement field components. In the problem considered, only the following components of the Christoffel symbols differ from zero: $\Gamma^2_{23} = \Gamma^2_{32} = 1/y^3$ and $\Gamma^3_{22} = -y^3/R^2$.

In the nonlinear theory [3–5], the elastic energy of the material is presented as an expansion with respect to strain-tensor invariants compatible with crystallographic symmetry of the medium. In an isotropic medium, there are only three independent invariants: $I_1 = E^m_m$, $I_2 = E^s_m E^m_s$, and $I_3 = E^n_m E^s_n E^m_s$; therefore, the general expression for the elastic energy of the shell has the form

$$U = \int\limits_{V_0} \varphi \sqrt{g}\; dy^1\, dy^2\, dy^3, \qquad \sqrt{g} = \sqrt{\det \|\boldsymbol{g}\|} = -y^3/R,$$

$$\varphi = \frac{\lambda}{2}\, I_1^2 + \mu I_2 + \frac{A}{3}\, I_3 + B I_1 I_2 + \frac{C}{3}\, I_1^3 + \sum_{n=4}^{\infty} \sum_{\langle k,p,q\rangle = n} A^{kpq} I_1^k I_2^p I_3^q.$$

Here φ is the energy normalized to the unit volume of the shell before its deformation, $\sum\limits_{\langle k,p,q\rangle = n}$ is the sum over all natural numbers k, p, and q with a restriction $k + 2p + 3q = n$. Integration is performed over the volume V_0 of the nondeformed shell. For certainty, we assume that all elastic moduli of the medium λ, μ, A, B, C, and A^{kpq} are comparable in order of magnitude.

Equations of dynamics of a nonlinearly elastic body are written as

$$-\rho_0\, \partial_t^2 v^i + \nabla_s P^{is} = 0, \tag{2}$$

where P^{is} are the components of the Piola–Kirchhoff tensor:

$$P^{is} = \frac{\partial \varphi}{\partial E_{is}} + \frac{\partial \varphi}{\partial E_{sm}}\, \nabla_m v^i. \tag{3}$$

For the further analysis, it is more convenient to write the boundary conditions on the part of the deformed body surface S, where the external forces $\boldsymbol{f} = f^k \boldsymbol{e}_k$ are defined, with respect to quantities corresponding to the nondeformed shell [3, 4]:

$$P^{ik} n_k\Big|_{\sigma'} = f^i\, \frac{dS}{d\sigma}\Big|_{\sigma'}.$$

Here n_l are the components of the normal vector to the corresponding element of the surface σ' of the nondeformed shell; $dS/d\sigma$ is the relative change in a small area during body deformation:

$$\frac{dS}{d\sigma} = \sqrt{m_k m^k}, \qquad m_k = \frac{\partial \det \|C\|}{\partial C_l^k}, \qquad C_l^k = \delta_l^k + \nabla_l v^k.$$

A specific feature of the hydrostatic pressure p is its permanent direction along the normal to the deformed surface. With allowance for this circumstance, the boundary conditions on the side surface σ of the shell acquire the form [3, 4]

$$P^{i3}\Big|_{\sigma^+} = g^{ik} p \,\frac{\partial \det \|C\|}{\partial C_3^k}\Big|_{\sigma^+}, \qquad P^{i3}\Big|_{\sigma^-} = 0. \tag{4}$$

Here σ^+ and σ^- are the external and internal parts of the side surface of the shell, respectively. The nonlinear boundary conditions (4) are responsible for the dispersion of local bendings of the shell.

At the same time, according to the Saint Venant principle, the description of dynamics of bendings in the central part of the shell requires only the integral characteristics of forces along the shell edges, which are usually taken into account by effective boundary conditions for simplified models [7].

2. Reductive Perturbation Theory. Let l be the characteristic scale of bendings of the shell surface ($l \ll L$, where L is the shell length) and $\tau_{\rm ch} = l/\sqrt{\mu/\rho_0}$ be the characteristic time of deformation. We introduce the following dimensionless variables:

$$\xi_\alpha = y^\alpha/l \quad (\alpha = 1, 2), \qquad \eta = (y^3 - R)/d, \qquad \tau = t/\tau_{\rm ch}.$$

In what follows, we consider the shell bendings commensurable with the shell thickness d; therefore, the displacement fields are normalized to d:

$$u = v^1/d, \qquad v = v^2/d, \qquad w = v^3/d.$$

We determine the small parameters characterizing the shell thickness and curvature: $\varepsilon = d/l \ll 1$ and $\delta = d/R = O(\varepsilon^2)$. We confine ourselves to considering rather slow processes for which the estimate $|\partial_\tau v^i/v^i| = O(\varepsilon^2)$ is valid. No more detailed information on the initial conditions of the problem is needed for a simplified model to be constructed.

Let the external pressure p be such that $p/\mu = O(\varepsilon^4)$. Under the conditions formulated above, the equations of the nonlinear theory of elasticity (2) for the shell can be reduced to a simpler model.

In dimensionless variables, the constitutive equations (2) have the form

$$\begin{gathered}\mu\varepsilon^2\,\partial_\tau^2 u = \varepsilon\,\partial_{\xi_\alpha} P^{1\alpha} + \partial_\eta P^{13} + \delta(1+\delta\eta)^{-1} P^{13},\\ \mu\varepsilon^2\,\partial_\tau^2 v = \varepsilon\,\partial_{\xi_\alpha} P^{2\alpha} + \partial_\eta P^{23} + \delta(1+\delta\eta)^{-1}(2P^{23} + P^{32}),\\ \mu\varepsilon^2\,\partial_\tau^2 w = \varepsilon\,\partial_{\xi_\alpha} P^{3\alpha} + \partial_\eta P^{33} - \delta(1+\delta\eta)^{-1} P^{22} + \delta(1+\delta\eta)^{-1} P^{33}.\end{gathered} \tag{5}$$

To construct the model, we seek for the solution of Eqs. (5) in the form

$$u = \sum_{n=1}^{\infty} u^{(n)}(\xi_1, \xi_2, \eta, \tau), \qquad v = \sum_{n=1}^{\infty} v^{(n)}(\xi_1, \xi_2, \eta, \tau), \qquad w = w^{(0)}(\xi_1, \xi_2, \tau) + \sum_{n=2}^{\infty} w^{(n)}(\xi_1, \xi_2, \eta, \tau). \tag{6}$$

Here the superscript determines the order of the term with respect to the small parameter ε. Formulas (1), (3), and (6) yield the expansions of the tensors E_{ij} and P^{ij}:

$$P^{ij} = \sum_n (P^{ij})^{(n)}, \qquad E_{ij} = \sum_n (E_{ij})^{(n)}. \tag{7}$$

Substituting formulas (7) into Eqs. (5) and equating the terms of an identical order of smallness with respect to the parameter ε, we obtain a chain of ordinary differential equations with respect to the "fast" variable η characterizing the nonuniformity of strains along the normal to the shell surface. The boundary conditions necessary for these equations to be solved follow from expansions of the initial boundary conditions (4) with respect to the parameter ε. In the first-order perturbation theory, the boundary-value problems

$$\partial_\eta (P^{\alpha 3})^{(s)} = 0, \qquad (P^{\alpha 3})^{(s)}\Big|_{\eta = \pm 1/2} = 0, \qquad s = 1, 2,$$

$$\partial_\eta (P^{33})^{(n)} = 0, \qquad (P^{33})^{(n)}\Big|_{\eta=\pm 1/2} = 0, \qquad n = 2,3 \tag{8}$$

have trivial solutions $(P^{\alpha 3})^{(s)} \equiv 0$ and $(P^{33})^{(n)} \equiv 0$. Using Eqs. (3), we can use these solutions to find the strains

$$E^{(s)}_{\alpha 3} = 0 \quad (s = 1,2), \qquad E^{(n)}_{33} = -\frac{\lambda}{\lambda + 2\mu}\, E^{(n)}_{\alpha\alpha} \quad (n = 2,3), \tag{9}$$

which allows us to simplify the expressions for the components $(P^{\alpha\beta})^{(n)}$ of the Piola–Kirchhoff tensor:

$$(P^{\alpha\beta})^{(n)} = \lambda' E^{(n)}_{\gamma\gamma} + 2\mu E^{(n)}_{\alpha\beta} \qquad (n = 2,3). \tag{10}$$

Hereinafter, the Greek indices take the values 1 and 2; the effective modulus $\lambda' = 2\mu\lambda/(\lambda + 2\mu)$ determines the stresses arising as the shell surface element changes its size.

If equalities (9) are written in terms of displacement fields, we obtain differential equations with respect to the variable η, which readily yield the dependence of the functions $u^{(s)}$, $v^{(s)}$, and $w^{(n)}$ on η. To construct the traditional nonlinear model of shells, it is sufficient to resolve only the first equations of system (8) with $s = 1$ with respect to displacements. As a result, we obtain

$$u^{(1)} = -\varepsilon\eta\, \partial_{\xi_1} \tilde{w}^{(0)} + \tilde{u}^{(1)}(\xi_1, \tau), \qquad v^{(1)} = -\varepsilon\eta\, \partial_{\xi_2} \tilde{w}^{(0)} + \tilde{v}^{(1)}(\xi_2, \tau). \tag{11}$$

The functions $\tilde{u}^{(1)}$ and $\tilde{v}^{(1)}$ independent of η, which appear in the course of integration, are rendered more concrete in the perturbation theory of the next orders. In what follows, we indicate all functions independent of η by the tilde, e.g., $w^{(0)} = \tilde{w}^{(0)}$.

Formulas (9)–(11) allow one to identify an explicit dependence of the functions $(P^{\alpha\beta})^{(2)}$ on the "fast" variable η:

$$(P^{\alpha\beta})^{(2)} = \sigma^{(2)}_{\alpha\beta} - \varepsilon^2 \eta (\lambda' \delta_{\alpha\beta} \Delta + 2\mu \partial_{\xi_\alpha} \partial_{\xi_\beta}) \tilde{w}^{(0)}, \qquad \Delta = \partial^2_{\xi_1} + \partial^2_{\xi_2} \tag{12}$$

($\delta_{\alpha\beta}$ is the Kronecker delta). The tensor $\sigma^{(2)}_{\alpha\beta} = \lambda' \varepsilon^{(2)}_{\gamma\gamma} \delta_{\alpha\beta} + 2\mu \varepsilon^{(2)}_{\alpha\beta}$ describes the internal stresses of the shell induced by "quasi-plane" strains:

$$\varepsilon^{(2)}_{11} = \varepsilon\, \partial_{\xi_1} \tilde{u}^{(1)} + \varepsilon^2 (\partial_{\xi_1} \tilde{w}^{(0)})^2/2, \qquad \varepsilon^{(2)}_{22} = \varepsilon\, \partial_{\xi_2} \tilde{v}^{(1)} + \varepsilon^2 (\partial_{\xi_2} \tilde{w}^{(0)})^2/2 + \delta \tilde{w}^{(0)},$$
$$\varepsilon^{(2)}_{12} = \varepsilon (\partial_{\xi_1} \tilde{v}^{(1)} + \partial_{\xi_2} \tilde{u}^{(1)} + \varepsilon\, \partial_{\xi_1} \tilde{w}^{(0)}\, \partial_{\xi_2} \tilde{w}^{(0)})/2. \tag{13}$$

It should be noted that the equations of the highest orders of the perturbation theory can be integrated only if the conditions of solvability of the corresponding boundary-value problems are satisfied. The conditions of solvability are obtained by integrating the equations of the perturbation theory over the shell thickness with allowance for the boundary conditions on the shell surface. These conditions yield algebraic or differential relations between functions, such as $\tilde{u}^{(1)}$ and $\tilde{v}^{(1)}$, which were arbitrary in the perturbation theory of the first orders. All solvability conditions can be demonstrated to be self-consistent and non-contradictory.

Let us determine the role of the solvability conditions by an example of a boundary-value problem of the third-order perturbation theory:

$$\partial_\eta (P^{\alpha 3})^{(3)} + \varepsilon\, \partial_{\xi_\beta} (P^{\alpha\beta})^{(2)} = 0, \qquad (P^{\alpha 3})^{(3)}\Big|_{\eta=\pm 1/2} = 0. \tag{14}$$

Integrating the equation in (14) over the shell thickness, we obtain the conditions of solvability of problem (14):

$$\partial_{\xi_\beta} \langle P^{\alpha\beta} \rangle^{(2)} = 0. \tag{15}$$

Hereinafter, $\langle f \rangle = \int\limits_{-1/2}^{1/2} f(\eta)\, d\eta$ is the mean value of the function $f(\eta)$ over the shell thickness. Using presentation (12), we can easily see that restrictions (15) reduce to a system of differential equations for calculating the functions $\tilde{u}^{(1)}$ and $\tilde{v}^{(1)}$:

$$\partial_{\xi_\beta} \sigma^{(2)}_{\alpha\beta} = 0. \tag{16}$$

If conditions (16) are satisfied, the boundary-value problem (14) has the following solution:

$$(P^{\alpha 3})^{(3)} = (\varepsilon^3/2)(\lambda' + 2\mu)(\eta^2 - 1/4) \Delta \partial_{\xi_\alpha} \tilde{w}^{(0)}.$$

Thus, in integrating the equations of the perturbation theory, we first calculate the functions $(P^{ik})^{(n)}$, which are related to the derivatives $(\partial\varphi/\partial E_{ik})^{(n)}$ by virtue of Eqs. (3) and then to the components of the strain tensor $E_{sp}^{(n)}$ via these derivatives. Therefore, with allowance for the equations of the perturbation theory of the previous orders, we calculate the functions $(\partial\varphi/\partial E_{ik})^{(n)}$ and $E_{sp}^{(n)}$ after determining $(P^{ik})^{(n)}$. This information is often sufficient for constructing simplified models of shells. There is no need to calculate all displacement fields up to the nth order inclusive. Constructing effective models is strongly facilitated by this feature of the perturbation theory.

Let us also note that $(P^{3\alpha})^{(m)} \neq (P^{\alpha 3})^{(m)}$ in the perturbation theory of the third order and higher. As it follows from Eqs. (3), however, the components of $(P^{3\alpha})^{(m)}$ can always be calculated from the known functions $(P^{\alpha 3})^{(m)}$. For example,

$$(P^{3\alpha})^{(3)} = (P^{\alpha 3})^{(3)} + \varepsilon\, \partial_{\xi_\gamma} \tilde{w}^{(0)} (P^{\gamma\alpha})^{(2)}.$$

The equation for calculating $\tilde{w}^{(0)}$ is obtained by using the boundary-value problem of the fourth-order perturbation theory:

$$\mu\varepsilon^2 \partial_\tau^2 \tilde{w}^{(0)} = \varepsilon \partial_{\xi_\alpha} (P^{3\alpha})^{(3)} + \partial_\eta (P^{33})^{(4)} - \delta (P^{22})^{(2)}, \quad (P^{33})^{(4)}\Big|_{\eta=1/2} = p, \quad (P^{33})^{(4)}\Big|_{\eta=-1/2} = 0. \tag{17}$$

We can easily verify that the conditions of solvability of problem (17) yield the differential equation

$$\mu\varepsilon^2 \partial_\tau \tilde{w}^{(0)} = -(1/12)(\lambda' + 2\mu)\varepsilon^4 \Delta\Delta \tilde{w}^{(0)} + p - \delta\sigma_{22}^{(2)} + \varepsilon^2 \sigma_{\gamma\beta}^{(2)}\, \partial_{\xi_\gamma}\, \partial_{\xi_\beta} \tilde{w}^{(0)}. \tag{18}$$

Equations (16) and (18) form a closed system for determining the functions $\tilde{u}^{(1)}$, $\tilde{v}^{(1)}$, and $\tilde{w}^{(0)}$, which coincides with the traditional nonlinear model of shells [1, 8, 9]. Equations (16) and (18) are normally written in another form, the displacement fields $\tilde{u}^{(1)}$ and $\tilde{v}^{(1)}$ being eliminated from Eqs. (13), (16), and (18) by using the stress function Φ: $\sigma_{11}^{(2)} = \partial_{\xi_2}^2 \Phi$, $\sigma_{22}^{(2)} = \partial_{\xi_1}^2 \Phi$, and $\sigma_{12}^{(2)} = -\partial_{\xi_1} \partial_{\xi_2} \Phi$.

Thus, the traditional nonlinear model of shells is a simplest reduction of the general equations of the elasticity theory (2). Such a model takes into account only the geometric nonlinearity of the problem and is not always suitable for studying the nonlinearly elastic dynamics of shells. Thus, the bendings along the generatrix of long hydrostatically compressed shells depend weakly on the coordinate ξ_1. At the same time, if we neglect the dependence of displacements on ξ_1 in the nonlinear equations (16) and (18), they reduce to linear equations. The nonlinear dynamics of shells compressed by a high-pressure fluid is manifested completely only in the reductive perturbation theory of the next orders and is caused not only by the geometric nonlinearity but also by the physical nonlinearity of the problem, and also by effects of the highest spatial dispersion.

To go outside the framework of the "quasi-linear" approximation (16), (18), constructing the model of shell bendings should involve the reductive perturbation theory of the next (fifth and sixth) orders. The procedure of constructing a more general dynamic model for a hydrostatically compressed shell is similar to the scheme whose details are given in [10] in the description of strong bendings of the plate.

The procedure of constructing effective models for shells and the models themselves can be substantially simplified by assuming that corrugation of shell surfaces near the thresholds of stability is performed by prevailing (neutrally stable) linear modes of deformation, which are specific for each particular problem. Because of instability of such modes, the nonlinear properties of the medium start manifesting and playing the governing role, which is responsible for spatial localization of strains. The effects of dispersion restrict the increase in amplitude of shell bendings and, under certain conditions, lead to formation of "long-lived" perturbations and patterns composed of dents on the shell surface. To describe local deformations of the shell theoretically, it is convenient to use the method of multiscale expansions, which reduces the equations of the nonlinear theory of elasticity to simplified amplitude equations for envelopes of the dents on the shell surface, which take into account the specific features of a particular problem. The use of this approach in previous investigations yielded an analytical description of corrugation of the most intensely loaded layer of the material [11], ring-shaped folds, and spatially localized patterns composed of diamond-shaped dents on the surfaces of longitudinally compressed cylindrical shells [12].

3. Amplitude Equation. Let us render expansion (6) more concrete with allowance for the geometry of bendings of the surface of a hydrostatically compressed shell. Let n waves of surface bending be formed along the arc of the shell. Then, the characteristic scale is $l = R/n$. As previously, we assume that $\varepsilon = d/l = nd/R \ll 1$ and $\delta = d/R = O(\varepsilon^2)$, which corresponds to formation of a large number of longitudinal dents on the shell surface: $n = O(1/\varepsilon)$, $n > 3$.

The shell deformations in the circumferential direction and along the normal to the surface are described by the previously used variables ξ_2 and η. Shell bendings along the generatrix are smooth; therefore, to analyze these bendings, we introduce a "slower" coordinate $X = \varepsilon y^1/l$. As $l/\varepsilon = R/(n\varepsilon) \sim R$, the variable X is appropriate for the description of shells whose length is greater than their radius. Further we consider rather slow processes described by the dimensionless time $T = \varepsilon^2 t/\tau_{\rm ch}$. The choice of scale transformations, which follows from the analysis of the space and time responses of the system to an external perturbation and allows taking into account the balance of the dispersion and nonlinearity effects [6], is commented below.

To construct the amplitude equation, we seek for the solution of the constitutive equations (5) in the form

$$u = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} u^{(n,m)}(X,T,\eta) \exp(im\xi_2), \qquad v = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} v^{(n,m)}(X,T,\eta) \exp(im\xi_2),$$

$$w = \tilde{w}^{(0,0)}(X,T) + \tilde{w}^{(0,1)}(X,T)\exp(i\xi_2) + \tilde{w}^{(0,-1)}(X,T)\exp(-i\xi_2) + \sum_{n=2}^{\infty} \sum_{m=-\infty}^{\infty} w^{(n,m)}(X,T,\eta)\exp(im\xi_2). \tag{19}$$

Here the index n characterizes the order of smallness of the term with respect to the parameter ε; the integer m determines the multiplicity of the harmonic. The relation between the wavenumber n/R of the prevailing neutrally stable mode and the external pressure is found in the course of model construction. The variables X and T describe slow modulations of the fundamental harmonic induced by its interaction with close unstable modes of deformation. By virtue of reality of the fields u, v, and w, the coefficients of expansions (19) satisfy the restrictions $u^{(n,m)} = (u^{(n,-m)})^*$, $v^{(n,m)} = (v^{(n,-m)})^*$, and $w^{(n,m)} = (w^{(n,-m)})^*$.

Let the shell ends be bound, so that the shell does not experience any longitudinal displacements:

$$\partial_X u^{(1,0)}\Big|_{y^1=0,L} = 0. \tag{20}$$

The equations of the perturbation theory are obtained by substituting expansions (19) into Eqs. (5) and the boundary conditions (4) after equating the terms of an identical order of smallness with respect to the parameter ε with identical multipliers $\exp(im\xi_2)$.

After integrating the equations of the perturbation theory with respect to the "fast" variable η, there arise functions of the slow variables X and T. From the conditions of solvability of equations, there also follows the relation between the wavenumber of the neutrally stable mode responsible for formation of shell bendings in the circumferential direction to the external pressure responsible for the beginning of changes in the shell form.

In constructing the simplified model, there is no need to calculate the displacement fields up to the sixth order inclusive. It is sufficient to find the dependence of the components $u^{(1,m)}$, $v^{(1,m)}$ $(m = 0,1,2)$, $u^{(2,s)}$, $v^{(2,s)}$ $(s = 0,1)$, $v^{(3,1)}$, and $w^{(2,p)}$ $(p = 0,1,2)$ on the variable η. The field components u and v are found by integrating the expressions for $(P^{\alpha 3})^{(s,m)}$ $(s = 1,2,3)$ with respect to η, and the corrections $w^{(2,p)}$ are found from equations $(P^{33})^{(2,p)} = 0$. The remaining information can be obtained from the functions $(P^{ij})^{(n,m)}$, which are calculated in a much easier way. In the second-order perturbation theory, only the following components of the tensors $E^{(2,m)}_{\alpha\beta}$ and $(P^{\alpha\beta})^{(2,m)}$ differ from zero:

$$E^{(2,0)}_{22} = \delta\tilde{w}^{(0,0)} + \varepsilon^2|\tilde{w}^{(0,1)}|^2, \qquad E^{(2,1)}_{21} = \varepsilon^2\eta\tilde{w}^{(0,1)},$$

$$(P^{11})^{(2,k)} = \lambda' E^{(2,k)}_{22}, \qquad (P^{22})^{(2,k)} = (\lambda' + 2\mu)E^{(2,k)}_{22}, \qquad k = 0,1. \tag{21}$$

We consider the specific features of calculations by an example of the equations of the fourth-order perturbation theory:

$$\varepsilon im(P^{32})^{(3,m)} + \partial_\eta(P^{33})^{(4,m)} - \delta(P^{22})^{(2,m)} = 0, (P^{33})^{(4,m)}\Big|_{\eta=1/2} = p\delta_{m,0}, \qquad (P^{33})^{(4,m)}\Big|_{\eta=-1/2} = 0. \tag{22}$$

Equations (22) of the perturbation theory of the previous orders reveal the dependence of all functions except for $(P^{33})^{(4,m)}$ on η.

For $m = 0$, the condition of solvability of problem (22) completely determines the component $(P^{22})^{(2,0)}$ as

$$\langle P^{22}\rangle^{(2,0)} \equiv (P^{22})^{(2,0)} = p/\delta \tag{23}$$

and also [as predicted by relations (21)] the functions $E^{(2,0)}_{22}$, $(P^{11})^{(2,0)}$, and $w^{(0,0)}$.

From the condition of solvability of problem (22) for $m = 1$, we obtain the relation between the wavenumber of the neutrally stable linear mode and the external pressure:

$$p = -\frac{\delta}{12}\,\varepsilon^2(\lambda' + 2\mu) \equiv -\frac{d^3 n^2}{12R^3}\,(\lambda' + 2\mu). \tag{24}$$

Formula (24) coincides with the definition of the "upper" critical load in the linear theory of shells [1]. For certainty, we further assume that the external pressure satisfies relation (24) with a relative error $O(\varepsilon^2)$. Then, we have $\langle P^{32}\rangle^{(3,1)} = O(\varepsilon^5)$.

Conditions (23) and (24) being satisfied, we can use Eqs. (22) to find the components $(P^{33})^{(4,m)}$ and also the functions $(\partial\varphi/\partial E_{33})^{(4,m)}$ and $E_{33}^{(4,m)}$, which are contained in the perturbation theory of the next orders.

We comment for the last step of constructing a simplified model without giving detailed calculations. The reductive perturbation theory is "closed" into a simplified model by solvability conditions, which have the following form $(s = 0, 1)$:

$$\mu\varepsilon^6\,\partial_T^2\tilde{w}^{(0,s)} = \varepsilon^2\,\partial_X\langle P^{31}\rangle^{(4,s)} + i\varepsilon\delta_{s1}\langle P^{32}\rangle^{(3,1)} + i\varepsilon s\langle P^{32}\rangle^{(5,s)}$$
$$- \delta\langle P^{22}\rangle^{(4,s)} - \delta^2\langle\eta P^{22}\rangle^{(2,s)} + \delta\langle P^{33}\rangle^{(4,s)} + (P^{33})^{(6,s)}\Big|_{\eta=1/2}. \tag{25}$$

Here $(P^{33})^{(6,0)}\Big|_{\eta=1/2} = p\delta\tilde{w}^{(0,0)}$ and $(P^{33})^{(6,1)}\Big|_{\eta=1/2} = p\varepsilon^2\tilde{w}^{(0,1)}/2$.

For $s = 0$, we determine the component $\langle P^{22}\rangle^{(4,0)}$ from Eq. (25). It should be noted that the explicit expression for $\langle P^{22}\rangle^{(4,0)}$ does not contain arbitrary functions, except for $\tilde{w}^{(0,1)}$. The procedure of calculating $\langle P^{22}\rangle^{(4,0)}$ is the same as that for $\langle P^{22}\rangle^{(2,0)}$ at the previous step.

For $s = 1$, the right side of equality (25) contains the term $\langle P^{32}\rangle^{(5,1)}$ whose calculation can be reduced to calculating the mean quantities $\langle P^{23}\rangle^{(5,1)}$, $\langle P^{22}\rangle^{(4,1)}$, $\langle P^{22}\rangle^{(4,2)}$, and $\langle P^{21}\rangle^{(3,0)}$ by the repeated use of formula (3). The mean quantities $\langle P^{22}\rangle^{(4,n)}$ $(n = 1, 2)$ are determined by the algebraic conditions of solvability of the previous boundary-value problems. For the function $\langle P^{21}\rangle^{(3,0)}$, the condition of solvability is a differential equation with respect to the variable X, which has a trivial solution $\langle P^{21}\rangle^{(3,0)} = 0$ under the boundary conditions (20). The function $(P^{23})^{(5,1)}$ is found by integrating the corresponding equation of the perturbation theory with respect to η. It should be noted that all mean quantities do not contain any arbitrary functions, except for $\tilde{w}^{(0,1)}$. For $s = 1$, the absence of the function $\tilde{w}^{(2,1)}$ in Eq. (25), which is contained in the perturbation theory of the next order, is guaranteed by the solvability condition (24) in the perturbation theory of the previous order.

Finally, the solvability condition (25) $(s = 1)$ yields a closed differential equation of the evolution of transverse bendings $\tilde{w}^{(0,1)}$ of a hydrostatically compressed shell

$$\partial_T^2\tilde{w}^{(0,1)} + a\tilde{w}^{(0,1)}\,\partial_T^2|\tilde{w}^{(0,1)}|^2 = b\,\partial_X^2\tilde{w}^{(0,1)} + c\tilde{w}^{(0,1)} - g\tilde{w}^{(0,1)}|\tilde{w}^{(0,1)}|^2, \tag{26}$$

where

$$a = \frac{\varepsilon^4}{\delta^2}, \qquad b = \frac{1}{3}\Big(1 + \frac{\lambda'}{2\mu}\Big), \qquad g = \frac{1}{4}\Big(1 + \frac{\lambda'}{2\mu}\Big),$$

$$c = -\frac{1}{\varepsilon^4\mu}\Big(\frac{p}{\delta} + \frac{\varepsilon^2}{12}\,(\lambda'+2\mu)\Big) + \frac{1}{6}\Big(1+\frac{\lambda'}{2\mu}\Big)\left[\frac{\delta^2}{\varepsilon^4} + \frac{1}{6}\left(\frac{17}{10} + \frac{1}{\lambda'+2\mu}\Big\{(A+2B)\Big[1-\Big(\frac{\lambda'}{2\mu}\Big)^3\Big] + (B+C)\Big(1-\frac{\lambda'}{2\mu}\Big)^3\Big\}\right)\right].$$

The simplified model (26) is suitable for describing the shell bendings everywhere except for narrow bands near the shell ends.

4. Patterns Composed of Dents and Compactons. The simplified model (26) has a wide class of exact solutions

$$\tilde{w}^{(0,1)} = A(X,T)\exp\,(i\Theta(X,T)),$$

which describe transverse displacements of the shell of the form

$$w = \frac{p}{\delta^2(\lambda'+2\mu)} - \frac{\varepsilon^2}{\delta}\,A^2(X,T) + 2A(X,T)\cos\,[\xi_2 + \Theta(X,T)]. \tag{27}$$

Here A and Θ are real functions.

A consequence of the amplitude equation (26) is the conservation law

$$\partial_T(\partial_T\Theta A^2) = b\,\partial_X(\partial_X\Theta A^2),$$

which may be satisfied with the use of the substitution

$$\Theta = \omega T + \varphi(X), \qquad A = A(X), \qquad \partial_X\varphi = r/A^2, \qquad r = \text{const.} \tag{28}$$

In the general case with $r \neq 0$, the bounded solutions of the model are written in terms of elliptic functions and integrals of the third kind. In a particular case, in choosing constants of integration, the solutions can be written in elementary functions:

$$A^2 = A_0^2(1 - \sin^2 B\,\text{sech}^2\,\Psi), \quad \Psi = \sqrt{g/(2b)}\,XA_0\sin B, \quad \varphi = \arctan\,(\tan\,B\tanh\,\Psi) + \sqrt{g/(2b)}\,XA_0\cos B. \tag{29}$$

Here B is a real parameter, $A_0^2 = 2\tilde{c}/[g(2+\cos^2 B)]$, and $\tilde{c} = c + \omega^2 > 0$.

For $\omega = 0$, formulas (27)–(29) describe a shell with dents extended along the generatrix, which are less expressed in a region with a width of the order of $l(\varepsilon A_0 \sin B)^{-1}\sqrt{2b/g}$, where the amplitudes of the bumps and dents in the shell cross section decrease by $2A_0 d\cos B$ beginning from $2A_0 d$ (in the original dimensional variables). For $\omega \neq 0$, a wave of transverse displacements travels over the shell surface with dents.

For $r = \omega = 0$ and $c > 0$, we have

$$A = \sqrt{2c/g}\,\sin B\,\text{sn}\,(X\sqrt{c/b}\,\cos B, k), \qquad \Theta = \text{const}, \tag{30}$$

where the real parameter $0 < B \leq \pi/4$ defines the amplitude and profile of the envelope of the shell bendings along the generatrix; $k = \tan\,B$ is the modulus of the elliptical sine. Solution (27), (30) corresponds to a motionless structure consisting of bumps and dents alternating along the shell generatrix and in the cross section of the shell. Apparently, such patterns of dents can be formed only if there are reinforcing rings inside the shell [1]. One half-wave of the envelope is normally formed along the shell generatrix. This situation is described by formulas (27), (30) with a parameter B, which is the root of the equation

$$(\varepsilon/l)L\sqrt{c/b}\,\cos B = 2K(k),$$

where $K(k)$ is the total elliptic integral of the first kind. The inequality $c > 0$ implies that the shell with dents can withstand a pressure whose absolute value is smaller than the critical load in the linear theory. As $B \to \pi/4$, the amplitude of the envelope is almost constant in the central part of the shell ($A(X) \simeq \sqrt{c/g}$) and is approximated by the following expression near the shell ends:

$$A(\xi) \simeq \sqrt{\frac{c}{g}}\,\tanh\,\Big(\frac{\varepsilon}{l}\sqrt{\frac{c}{2b}}\,\xi\Big), \qquad 0 \leq \xi \leq \frac{l}{\varepsilon}\sqrt{\frac{2b}{c}}$$

($\xi = y^1$ or $\xi = L - y^1$). The fact that the bendings along the generatrix of long shells are "leveled off" is observed experimentally and cannot be explained by the traditional theory of shells.

Model (26) admits propagation of nonlinear monochromatic waves along a shell with dents. The corresponding transverse displacements of the shell are described by Eq. (27) where one has to assume that

$$A(X,T) = A_0\sin\,(\Omega T + kX), \quad \Theta = 0, \quad \Omega = \frac{1}{2}\sqrt{\frac{g}{a}}, \quad k^2 = \frac{g}{4ab}\,(1 - 2aA_0^2) + \frac{c}{b} > 0.$$

Here A_0 is a real parameter.

After the substitution $\tilde{w}^{(0,1)} = A(\zeta)$, $\zeta = X \pm \sqrt{b}T$ and integration with respect to ζ (the integration constant is chosen to be zero), the amplitude equation (26) acquires the form

$$A^2\Big((\partial_\zeta A)^2 + \frac{g}{4ab}\,A^2 - \frac{c}{2ab}\Big) = 0. \tag{31}$$

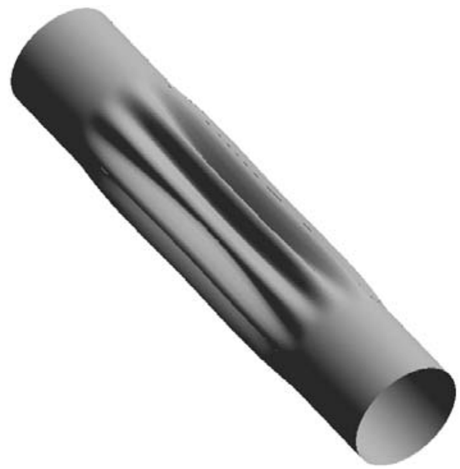

Fig. 1. Compacton corresponding to solution (32).

Among the solutions of Eq. (31), there are exotic solitons, namely, compactons [13]:

$$A(\zeta) = \sqrt{\frac{2c}{g}}\,\cos\Bigl(\frac{1}{2}\sqrt{\frac{g}{ab}}\,\zeta\Bigr), \qquad |\zeta| \le \pi\sqrt{\frac{ab}{g}}. \tag{32}$$

Outside the indicated interval, the function $A(\zeta)$ vanishes. Though the derivatives of this function are discontinuous at the compacton ends, the terms with the derivatives $A^2(\partial_\zeta A)^2$ and $A\,\partial_\zeta^2 A^2$ in Eqs. (26) and (31) are continuous everywhere including the soliton edges. The transverse bendings of the shell corresponding to a compacton are shown in Fig. 1. By virtue of their features, compactons do not interact with other solitons before their collision. They move along the shell generatrix with a velocity $l\varepsilon\sqrt{b}/\tau_{\text{ch}}$ (in the original dimensional variables). As compactons are formed only near the threshold of stability of the shell form, they can be used to diagnose the pre-critical state of the shell.

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